Modeling Uncertainty - 1

Decision Supports Systems 2017/18, Lecture 04

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Modeling Uncertainty

- uncertainty is a critical element of many decisions

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- how do we model uncertainty?
  - probability

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- how do we model uncertainty?
  - probability
- sometimes, the problem at hand is similar to some **prototypical situations**
  - we will look at some such standard models

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Basics of Probability

- a central principle in decision making is that we can **represent uncertainty through the appropriate use of probability**
- many uncertain events are quantitative (e.g. tomorrow's max temperature)
- if not quantitative, we can introduce a quantitative variable:
  - $X = 1$ if it rains
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Basics of Probability

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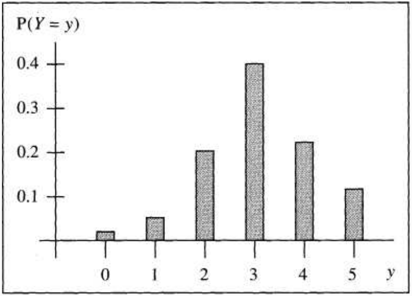
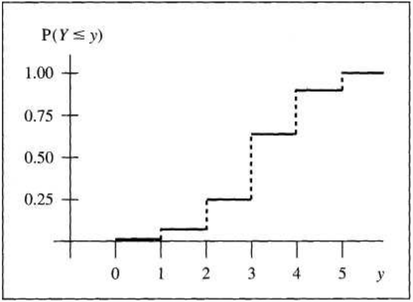
- the set of probabilities associated with all possible outcomes = **probability distribution**
- example:
  - we denote the number of raisins in an oatmeal cookie as Y
  - $P(Y = 0) = 0.02$
  - $P(Y = 1) = 0.05$
  - $P(Y = 2) = 0.20$
  - etc
- all probabilities ina probability distribution sum up to 1
- uncertain quantities (e.g. number of raisins Y) are called **random variables**

Discrete Probability Distributions

- the uncertain quantity can assume a **finite** or countable number of possible values
- example:
  - raisins in oatmeal cookie
  - precipitation (yes/no)
- we describe it with
  - probability mass function (PMF)
  - cumulative distribution function (CDF)



the probability that a discrete random variable Y is exactly equal to some value y



the probability that Y will take a value less than or equal to some value y

Expected value

- the **expected value** of a discrete random variable X is its probability-weighted average
- also average, mean, μ

$$E(X) = \sum_{i=1}^{n} x_i \cdot P(X = x)$$

- best guess

Variance and Standard Deviation

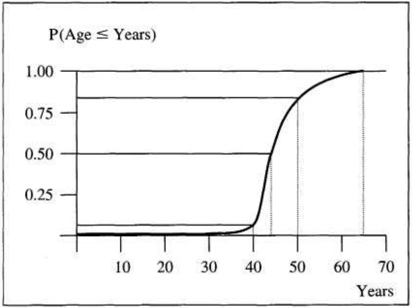
- variance = sum of squares of deviations from mean
- also $Var(X)$ or σ_X^2

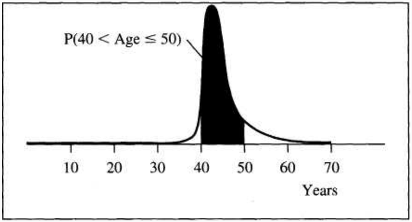
$$Var(X) = \sum_{i=1}^{n} [x_i - E(X)]^2 \cdot P(X = x)$$

- standard deviation = square root of variance
- also σ_X
- best guess of how far the outcome might lie from $E(X)$
- a large variance or standard deviation indicates that the outcome is highly variable and hard to predict

Continuous Probability Distributions

- the uncertain quantity (represented by the random variable X) can take a value within a range (as opposed to discrete distributions)
- example
  - tomorrow max temperature
- we typically speak about interval probabilities $P(a \leq Y \leq b)$
- we describe it with
  - probability density function (PDF)
  - cumulative distribution function (CDF)





Statistical Moments

- expected value

$$E(X) = \int_{x-}^{x+} x f(x) dx$$

- variance

$$Var(X) = \sigma_X^2 = \int_{x-}^{x+} [x - E(X)]^2 f(x) dx$$

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Theoretical Probability Models

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- . . . we can find a similar existing protoypical distribution

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- it is important to have a good overview of the main distributions and their natural applications

Distributions

- there is A LOT of distributions
- we will cover the following distributions
  - discrete
    - binomial distribution
    - Poisson distribution
  - continuous
    - exponential distribution
    - normal distribution
    - beta distributions
- for each distribution:
  - typical example
  - parameters
  - shape
  - no equations in this course

Binomial Distribution

- discrete distribution
- example
  - you are in a race for mayor of your hometown, and
  - you wanted to find out how you were doing with the voters.
  - you might take a sample and count the number of individuals who indicated a preference for you
  - in this situation, each voter interviewed can be either for you or not.

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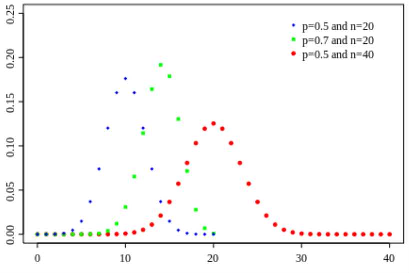
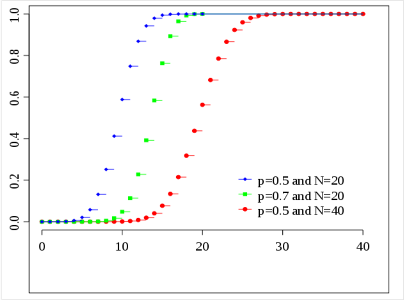
- Binomial distribution is applicable in cases where:
    - outcomes are *dichotomous*: sequential, each can be only true/false
    - *constant probability p* at each event (trial) the probability of true is always the same
    - the outcome of each trial is *independent* of the previous ones

Binomial Distribution

- parameters of the binomial distribution $B(n, p)$ are
  - $n \in N_0$ — number of trials
  - $p \in [0, 1]$ — success probability in each trial
- when to use?
  - The binomial distribution is frequently used to model the number of successes in a sample of size n drawn with replacement from a population of size N.
  - e.g. when having N voters, what is the chance that n will vote for you if the probability of voting is p?





Poisson distribution

- discrete distribution
- representing occurrences of a particular event over time or space
- example
  - you are interested in the number of customers who arrive at a bank in one hour.
  - this is an uncertain quantity; there could be none, one, two, three, etc.

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- The Poisson distribution is an appropriate model if the following assumptions are true.
  - k is the number of times an event occurs in an interval and k can take values 0, 1, 2, ...
  - The occurrence of one event does not affect the probability that a second event will occur. That is, events occur independently.
  - The rate at which events occur is constant. The rate cannot be higher in some intervals and lower in other intervals.
  - Two events cannot occur at exactly the same instant; instead, at each very small sub-interval exactly one event either occurs or does not occur.
  - The probability of an event in a small sub-interval is proportional to the length of the sub-interval.

Poisson distribution

- parameters of the distribution $P(\lambda)$
  - $\lambda > 0$ real

Poisson distribution

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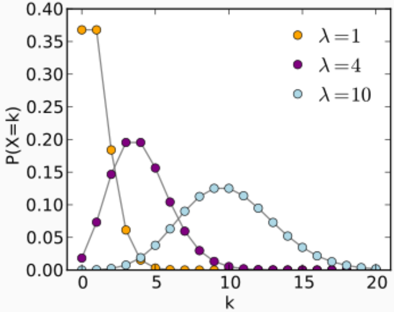
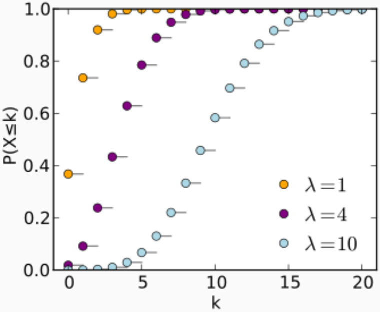
- when to use?
  - for modelling the number of times an event occurs in an interval of time or space.
  - e.g. what is the chance that k people will walk into the bank between 10am and 11am?





x axis: k . . . number of occurrences

Exponential distribution

- continuous distribution
- also known as negative exponential distribution
- related to the Poisson distribution
  - describes the time between events in a Poisson process
- example:
  - Poisson: number of arrivals in a time window
  - exponential: time between arrivals

Exponential distribution

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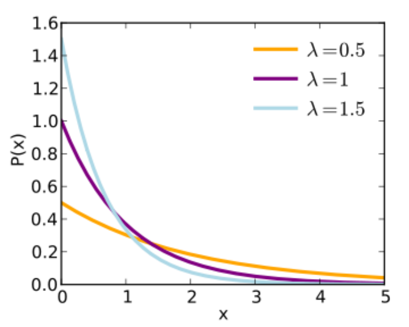
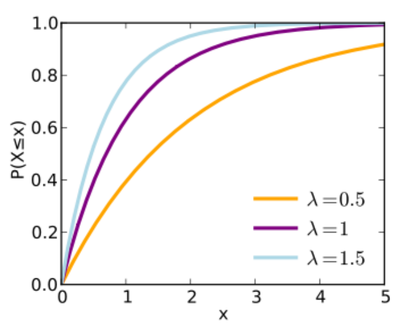
- related to the Poisson distribution
  - describes the time between events in a Poisson process
- example:
  - Poisson: number of arrivals in a time window
  - exponential: time between arrivals
- has the same requirements as the Poisson distribution

Exponential distribution

- parameters :
  - $\lambda > 0$ real



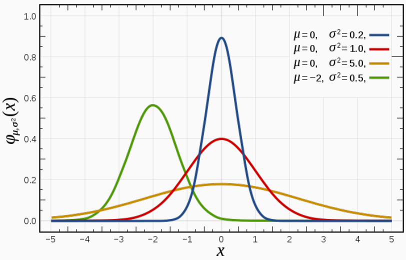


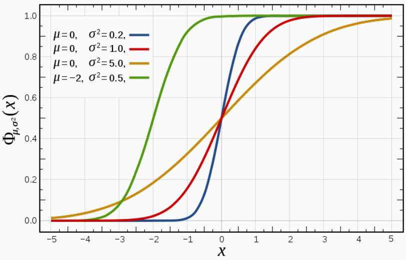
Normal Distribution

- continuous
- when uncertainty is due to several sources of uncertainty
- example:
  - measurement errors are due to environmental conditions, equipment malfunctions, human error, etc.
- central limit theorem:
  - averages of samples of observations of random variables independently drawn from independent distributions converge in distribution to the normal when the number of observations is sufficiently large.

Normal Distribution

- parameters of the $\mathcal{N}(\mu, \sigma^2)$
  - μ - mean
  - $\sigma^2 > 0$ - variance
- rules of thumb:
  - $P \approx 0.68$ that a normal random variable is within one standard deviation of the mean
  - $P \approx 0.95$ that it is within two standard deviations of the mean.



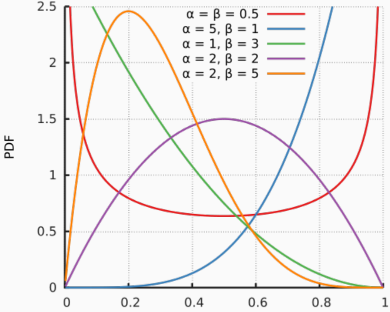


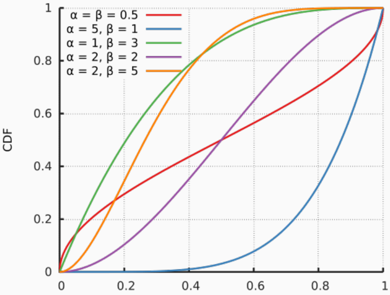
Beta Distribution

- a family of continuous probability distributions defined on the interval $[0, 1]$
- applicable when a random variable is in a limited interval
- example:
    - the proportion of people who will vote for candidate A

Beta Distribution

- parameters of *Beta*(α, β)
  - $\alpha > 0$ shape
  - $\beta > 0$ shape





Quiz

Which distributions are good choices for the following examples of uncertainty

1. Educational and intelligence tests [normal]
2. In many market research studies, a fundamental issue is whether a potential customer prefers one product to another [binomial]
3. How many defects are acceptable in a finished product? In some products, the occurrence of defects, such as bubbles in glass or blemishes in cloth happens from time to time. [Poisson, exponential]
4. How to provide adequate service (e.g. how many cashiers should be open) when the arrival of customers is uncertain. [Poisson, exponential]

References

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- https://en.wikipedia.org/wiki/Poisson_distribution
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- https://en.wikipedia.org/wiki/Normal_distribution
- https://en.wikipedia.org/wiki/Beta_distribution